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ACTIVE STABILIZATION OF THE ROTARY MOTION OF A SOLID BODY

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A method for constructing control moments for application to a solid body rotating about a fixed point [1] is proposed. The effect of these moments is to stabilize an axis rigidly attached to the solid body with respect to some other axis passing through the fixed point and performing a specified motion. The problem of stabilization of the rigidly attached axis with respect to a stationary axis passing through a fixed point is considered in the case of a girostat.

1. Let us consider a solid body with a fixed point O which coincides with the body center of mass. We denote by $\mathbf{S}$ a vector in the absolute system of coordinates $oXYZ$; by $\mathbf{H}$ a vector rigidly attached to the solid body; by $oxyz$ a system of coordinates whose axes coincide with the principal central axes of the body ellipsoid of inertia, and by $\mathbf{S}_0$ and $\mathbf{H}_0$ the unit vectors of vectors $\mathbf{S}$ and $\mathbf{H}$. Vector $\mathbf{S}$ is assumed to rotate about point o at some instantaneous velocity $\boldsymbol{\omega}_0(t)$. We use the notation

$$\alpha_1 = \cos(\mathbf{S}_0, x), \quad \alpha_2 = \cos(\mathbf{S}_0, y), \quad \alpha_3 = \cos(\mathbf{S}_0, z)$$
$$\beta_1 = \cos(\mathbf{H}_0, x), \quad \beta_2 = \cos(\mathbf{H}_0, y), \quad \beta_3 = \cos(\mathbf{H}_0, z)$$

We write the equations of motion in the form of Euler's dynamic equation

$$\theta\boldsymbol{\omega}^{\cdot} + \boldsymbol{\omega} \times \Theta\boldsymbol{\omega} = \mathbf{M} \tag{1.1}$$

where $\boldsymbol{\omega}$ is the vector of instantaneous angular velocity of the body, θ is the tensor of inertia of that body about the fixed point o, and $\mathbf{M}$ is the control moment. Vector $\mathbf{S}_0$ satisfies the kinematic relation

$$\mathbf{S}_0^{\cdot} = \mathbf{S}_0 \times (\boldsymbol{\omega} - \boldsymbol{\omega}_0) \tag{1.2}$$

Let us consider the problem of determining a control moment $\mathbf{M}$ whose action on the body would force it to approach asymptotically a motion such that vector $\mathbf{H}$ attached to the body assumed the direction of the mobile vector $\mathbf{S}$, and to investigate the stability of such motion.

Let us consider moment $\mathbf{M}$ analogous to that considered in [1] defined by

$$\mathbf{M} = \boldsymbol{\mu} + \Theta\boldsymbol{\omega}_0^{\cdot} + \boldsymbol{\omega}_0 \times \Theta\boldsymbol{\omega} + \lambda(\mathbf{H}_0 \times \Theta\boldsymbol{\omega}) + {}^1/_2\,\mathrm{grad}_{\alpha_i} U, \quad \lambda = \mathrm{const} > 0 \tag{1.3}$$

$$U = \alpha \sum_{i=1}^{3} (\alpha_i - \beta_i)^2, \quad \alpha = \mathrm{const} > 0 \tag{1.4}$$

where $\mathrm{grad}_{\alpha_i} U$ indicates that the gradient operation is carried out with respect to components α_i $(i = 1, 2, 3)$. The equations of motion (1.1) – (1.3) admit the following particular solution

$$\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \lambda\mathbf{H}_0, \quad \mathbf{H}_0 = \mathbf{S}_0 \tag{1.5}$$

where $\boldsymbol{\mu}$ is some moment which vanishes in the case of motion (1.5).

Let us investigate the stability of solution (1.5). We form the equations of perturbed motion, using for the variations of variables the following notation:

$$p - p_0 - \lambda\beta_1 = p_1, \quad q - q_0 - \lambda\beta_2 = q_1, \quad r - r_0 - \lambda\beta_3 = r_1, \quad \alpha_i - \beta_i = \delta_i \quad (i = 1, 2, 3) \tag{1.6}$$

We have

$$Ap_1^{\bullet} = \lambda\,(B\beta_2 r_1 - C\beta_3 q_1) + Bq_0 r_1 - Cr_0 q_1 + (B - C)q_1 r_1 + \alpha\delta_1 + \mu_1 \quad (1\ 2\ 3) \tag{1.7}$$

$$\delta_1^{\bullet} = \beta_2 r_1 - \beta_3 q_1 + \lambda\,(\beta_3\delta_2 - \beta_2\delta_3) + \delta_2 r_1 - \delta_3 q_1 \quad (1\ 2\ 3) \tag{1.8}$$

where the symbol (123) shows that the two subsequent equations are obtained from (1.7) and (1.8) by cyclic permutation.

Let us consider function

$$2V = Ap_1^2 + Bq_1^2 + Cr_1^2 - 2\lambda\,(Ap_1\delta_1 + Bq_1\delta_2 + Cr_1\delta_3) + \alpha\,(\delta_1^2 + \delta_2^2 + \delta_3^2) \tag{1.9}$$

for

$$\alpha > \lambda^2 m_1, \quad m_1 = \max\,\{A,\ B,\ C\} \tag{1.10}$$

Function (1.9) is a positive definite function of variations of the system variables. Let us formulate the total derivative of function (1.9) with respect to time on the basis of equations of perturbed motion (1.7),(1.8) and assume

$$\begin{aligned}\mu_1 &= -Kp_1 - [K - \lambda\,(B + C)]\,(\beta_3 q_1 - \beta_2 r_1) - (Bq_0 r_1 - Cr_0 q_1)\\ \mu_2 &= -Kq_1 - [K - \lambda\,(A + C)]\,(\beta_1 r_1 - \beta_3 p_1) - (Cr_0 p_1 - Ap_0 r_1)\\ \mu_3 &= -Kr_1 - [K - \lambda\,(A + B)]\,(\beta_2 p_1 - \beta_1 q_1) - (Ap_0 q_1 - Bq_0 p_1), \quad K = \alpha/\lambda > 0\end{aligned} \tag{1.11}$$

We obtain

$$V^{\bullet} = -K\,[(p_1 - \lambda\delta_1)^2 + (q_1 - \lambda\delta_2)^2 + (r_1 - \lambda\delta_3)^2] \tag{1.12}$$

Function (1.12) is a constant-sign negative function of variables $p_1, q_1, r_1, \delta_1, \delta_2$ and δ_3, and the manifold E of points at which $V^{\bullet} = 0$ is of the form

$$p_1 = \lambda\delta_1, \quad q_1 = \lambda\delta_2, \quad r_1 = \lambda\delta_3 \tag{1.13}$$

For parameters defined by (1.13) the equations of motion (1.8) are satisfied identically and Eqs. (1.7) assume the form

$$\lambda\,(B - C)\delta_2\delta_3 + (K - \lambda C)\,\beta_2\delta_3 - (K - \lambda B)\beta_3\delta_2 = 0 \quad (1\ \ 2\ \ 3) \tag{1.14}$$

Note that when δ_i $(i = 1,\ 2,\ 3)$ is determined with the use of one of the equations of system (1.14) and the result substituted into another, the third of these equations is obtained, hence one of the equations of that system can be excluded from the analysis. From the first of Eqs. (1.14) we determine δ_2, and from the second δ_1. The substitution of these into formula

$$\delta_1^2 + \delta_2^2 + \delta_3^2 + 2\,(\beta_1\delta_1 + \beta_2\delta_2 + \beta_3\delta_3) = 0 \tag{1.15}$$

yields for δ_3 the following equation:

$$f\,(\delta_3) = (K - \lambda C)^2\,\beta_1^2\delta_3\Delta_1^2 + (K - \lambda C)^2\beta_2^2\delta_3\Delta_2^2 + 2\,(K - \lambda C)\,\beta_1^2\Delta_1 + 2\,(K - \lambda C)\,\beta_2^2\Delta_2 + \delta_3 + 2\beta_3 = 0 \tag{1.16}$$

$$\Delta_1^{-1} = [(K - \lambda A)\,\beta_3 - \lambda\,(A - C)\,\delta_3], \quad \Delta_2^{-1} = [(K - \lambda B)\beta_3 - \lambda\,(B - C)\,\delta_3]$$

Function $f(\delta_3)$ changes its sign along segment $[-1-\beta_3,\ 1-\beta_3]$.

We denote the root of Eq. (1.16) by l_3 and substituting the latter into Eqs. (1.14), we determine $\delta_1 = l_1$ and $\delta_2 = l_2$. After this the manifold (1.13) will not contain any other complete motions, except the unperturbed motion for which

$$p_1 = q_1 = r_1 = 0, \quad \delta_1 = \delta_2 = \delta_3 = 0 \tag{1.17}$$

provided the initial perturbations belong to region

$$p_{10}^2 + q_{10}^2 + r_{10}^2 < \lambda^2 l^2, \quad \delta_{10}^2 + \delta_{20}^2 + \delta_{30}^2 < l^2, \quad l^2 = l_1^2 + l_2^2 + l_3^2 \tag{1.18}$$

Thus, when equalities (1.11) are satisfied, the unperturbed motion (1.17) is asymptotically stable for all initial perturbations from region (1.18) [2].

Theorem 1. If initial perturbations belong to region (1.18), then with condition (1.10) and equalities (1.11) satisfied, the solid body subjected to the action of moment

$$\mathbf{M} = \boldsymbol{\mu} + \Theta\boldsymbol{\omega}_0^{\cdot} + \boldsymbol{\omega}_0 \times \Theta\boldsymbol{\omega} + \lambda(\mathbf{H}_0 \times \Theta\boldsymbol{\omega}) + {}^1/_2\,\mathrm{grad}_{\alpha_i} U \tag{1.19}$$

either performs motion

$$\boldsymbol{\omega} = \lambda\mathbf{H}_0 + \boldsymbol{\omega}_0, \quad \mathbf{H}_0 = \mathbf{S}_0 \tag{1.20}$$

or tends asymptotically to such motion; according to Liapunov, motion (1.20) is stable.

Corollary 1. If vector $\mathbf{H}$ lies on one of the semiaxes of the body ellipsoid of inertia and the number K is determined in conformity with the inequality

$$\frac{K - \lambda m_1}{\lambda(m_1 - m_2)} > 2, \quad m_2 = \min\{A, B, C\} \tag{1.21}$$

the motion defined by (1.20) is asymptotically stable for all initial perturbations from the region

$$p_{10}^2 + q_{10}^2 + r_{10}^2 < 4\lambda^2, \quad \delta_{10}^2 + \delta_{20}^2 + \delta_{30}^2 < 4 \tag{1.22}$$

If vector $\mathbf{H}$ lies on one of the semiaxes of the body ellipsoid of inertia and the term $\lambda(\mathbf{H}_0 \times \Theta\boldsymbol{\omega})$ is absent from formula (1.19) for the moment, then, as shown by Zubov [1], the motion (1.20), for certain conditions imposed on moment $\boldsymbol{\mu}$, is conditionally stable by Liapunov's definition.

Now, let $\mathbf{S}$ be a fixed vector in the absolute system of coordinates $oXYZ$.

Corollary 2. If the initial perturbations belong to region (1.18) and condition (1.10) and equalities (1.11) $(p_0 = q_0 = r_0 = 0)$ are satisfied, the solid body subjected to the action of moment

$$\mathbf{M} = \boldsymbol{\mu} + \lambda(\mathbf{H}_0 \times \Theta\boldsymbol{\omega}) + {}^1/_2\ \mathrm{grad}_{\alpha_i} U \tag{1.23}$$

either performs motion

$$\boldsymbol{\omega} = \lambda\mathbf{H}_0, \quad \mathbf{H}_0 = \mathbf{S}_0 \tag{1.24}$$

or tends asymptotically to such motion and, according to Liapunov, (1.24) is asymptotically stable.

Let number K be defined by (1.21) and vector $\mathbf{H}$ lie on one of the semiaxes of the body ellipsoid of inertia. Then the constant rotations of the body about the smallest $(A > B > C)$ and the greatest $(A < B < C)$ semiaxes of the body ellipsoid of inertia are asymptotically stable for all perturbations from region (1.22).

The constant rotation about the middle semiaxis of the body ellipsoid of inertia is asymptotically stable for all initial perturbations from region (1.22), if $(A > B > C)$

and the number K satisfies condition

$$K > \max\{\lambda(2B - C),\ \lambda(3A - 2B)\}$$

If moment μ is absent from formula (1.23), the unperturbed motion (1.17) is unstable, since the characteristic equation of the system of the first approximation equations (1.7) and (1.8) of perturbed motion ($p_0 = q_0 = r_0 = 0$) has in addition to the zero root at least one positive root

$$\Delta(\zeta) = \zeta\varphi(\zeta) = 0$$

where

$$\varphi(0) = -\lambda^4 K[\beta_1^2\beta_2^2(A - B)^2 + \beta_1^2\beta_3^2(A - C)^2 + \beta_2^2\beta_3^2(C - B)^2] < 0, \qquad \varphi(\zeta \to +\infty) \to +\infty$$

Below we assume that $\mathbf{S}$ is a fixed vector in the absolute system of coordinates $oXYZ$, i,e, that $\omega_0 = 0$.

2. Let us consider the motion of a solid body in a central Newtonian force field. Let the fixed point O of the body be fixed at distance R from the center of pull O_1. The axis oZ of the fixed system of coordinates $oXYZ$ is directed outward from the center of pull and $\mathbf{S}_0$ is the unit vector of that axis. The equations of motion of the body with allowance for moments of gravitational forces are of the form

$$\Theta\omega^{\cdot} + \omega \times \Theta\omega - \nu(\mathbf{S}_0 \times \Theta\mathbf{S}_0) = \mathbf{M}, \quad \mathbf{S}_0^{\cdot} = \mathbf{S}_0 \times \omega, \quad \nu = 3g_0/R \qquad (2.1)$$

where g_0 is the acceleration of gravity at distance R.

Let us analyze the moment

$$\mathbf{M} = \mu^\circ + \lambda(\mathbf{H}_0 \times \Theta\omega) - \nu(\mathbf{H}_0 \times \Theta\mathbf{S}_0) + {}^1/_2\,\mathrm{grad}_{\alpha_i} U \qquad (2.2)$$

The equations of motion (2.1) and (2.2) admit the particular solution

$$\omega = \lambda\mathbf{H}_0, \quad \mathbf{H}_0 = \mathbf{S}_0 \qquad (2.3)$$

Let us investigate the stability of solution (2.3). Retaining the notation used in (1.6) ($p_0 = q_0 = r_0 = 0$) we formulate the equations of perturbed motion

$$Ap_1^{\cdot} = \lambda(B\beta_2 r_1 - C\beta_3 q_1) + (B - C)q_1 r_1 + \nu(C - B)\delta_2\delta_3 + \nu(C\beta_3\delta_2 - B\beta_2\delta_3) + \alpha\delta_1 + \mu_1^\circ \quad (1\ 2\ 3) \qquad (2.4)$$

$$\delta_1^{\cdot} = \beta_2 r_1 - \beta_3 q_1 + \lambda(\beta_3\delta_2 - \beta_2\delta_3) + r_1\delta_2 - q_1\delta_3 \quad (1\ \ 2\ \ 3) \qquad (2.5)$$

Function

$$2V = Ap_1^2 + Bq_1^2 + Cr_1^2 - 2\lambda(Ap_1\delta_1 + Bq_1\delta_2 + Cr_1\delta_3) + (\alpha + \nu A)\delta_1^2 + (\alpha + \nu B)\delta_2^2 + (\alpha + \nu C)\delta_3^2 \qquad (2.6)$$

where

$$\alpha > (\lambda^2 - \nu)m_1, \quad m_1 = \max\{A,\ B,\ C\} \qquad (2.7)$$

is positive-definite with respect to variables appearing in it.

Let us set

$$\mu_1^0 = \mu_1 - \nu(B + C)(\beta_3\delta_2 - \beta_2\delta_3), \quad \mu_2^\circ = \mu_2 - \nu(A + C)(\beta_1\delta_3 - \beta_3\delta_1) \qquad (2.8)$$

$$\mu_3^0 = \mu_3 - \nu(A + B)(\beta_2\delta_1 - \beta_1\delta_2)$$

where μ_i are functions defined in (1.11) ($p_0 = q_0 = r_0 = 0$). The derivative of function (2.6) with respect to time derived on the basis of equations of perturbed motions (2.4) and (2.5) with allowance for (1.11) and (2.8) are of the form (1.12).

We set $\lambda^2 = \nu$. Then for $p_1 = \lambda\delta_1$, $q_1 = \lambda\delta_2$, and $r_1 = \lambda\delta_3$ Eqs. (2.4) assume the form $\beta_2\delta_3 - \beta_3\delta_2 = 0$, $\beta_3\delta_1 - \beta_1\delta_3 = 0$, $\beta_1\delta_2 - \beta_2\delta_1 = 0$.

Hence the manifold (1.14) in which $V^{\cdot} = 0$ does not contain any other complete motions of the system, except the unperturbed motion (1.17), if the initial perturbations belong to region (1.22).

Theorem 2. If $\lambda^2 = \nu$, the initial perturbations belong to region (1.22) and the equalities (1.11) and (2.8) are satisfied, the solid body subjected to moment (2.2) either performs motion (2.3) or tends asymptotically to that motion and such motion is asymptotically stable according to Liapunov.

3. Let us consider a solid body with a fixed point on whose principal axes of inertia lie three axes of similar symmetric flywheels. Such system belongs to the class of gyrostats, i.e. of system whose mass distribution remains unaltered during motion. The law of the moment of momentum yields the following equations:

$$C_1p^{\cdot} + J_1\Omega_1^{\cdot} + (C_3 - C_2)\, qr + H_3q - H_2r = 0 \quad (1\ 2\ 3), \quad H_i = J_i\Omega_i \quad (i = 1, 2, 3) \tag{3.1}$$

where C_i are the principal central moments of inertia of the gyrostat; J_i are the related moments of inertia of flywheels; Ω_i are the angular velocities of flywheel rotation relative to the body, and p, q and r are projections of vector ω of the body angular velocity on the xyz-axes.

The equations of motion of the flywheels are

$$J_1\,(\Omega_1^{\cdot} + p^{\cdot}) = -\mathrm{M}_x, \quad J_2\,(\Omega_2^{\cdot} + q^{\cdot}) = -\mathrm{M}_y, \quad J_3\,(\Omega_3^{\cdot} + r^{\cdot}) = -\mathrm{M}_z \tag{3.2}$$

where $-\mathrm{M}_x$, $-\mathrm{M}_y$ and $-\mathrm{M}_z$ represent the torques of motors driving the flywheels.
Let

$$A = C_1 - J_1, \quad B = C_2 - J_2, \quad C = C_3 - J_3$$

From Eqs. (3.1) and (3.2) we obtain

$$Ap^{\cdot} = (C_2q + H_2)\, r - (C_3r + H_3)\, q + \mathrm{M}_x \quad (1\ \ 2\ \ 3) \tag{3.3}$$

Equations (3.3) are taken as the basis of investigation, and moments M_x, M_y and M_z are considered to be the control moments for the body.

We introduce the new variables

$$z_1 = Ap + J_1\,(\Omega_1 + p), \quad z_2 = Bq + J_2\,(\Omega_2 + q), \quad z_3 = Cr + J_3\,(\Omega_3 + r) \tag{3.4}$$

Then the combined Eqs. (3.2) and (3.3) assume the form

$$Ap^{\cdot} = z_2r - z_3q + \mathrm{M}_x \quad (1\ \ 2\ \ 3) \tag{3.5}$$

$$z_1^{\cdot} = z_2r - z_3q \quad (1\ \ 2\ \ 3) \tag{3.6}$$

Equations (3.6) admit the first integral

$$z_1^2 + z_2^2 + z_3^2 = \text{const} \tag{3.7}$$

Hence for $z_1^2\,(0) + z_2^2\,(0) + z_3^2\,(0) < \infty$ functions z_i are always bounded. The existence of the integral (3.7) shows that it is impossible to simultaneously reduce all variables to zero. This conclusion shows the futility of attempts at stabilizing the gyrostat, and the necessity to confine the problem to that of stabilizing the rotation of the body itself [3].

Let us associate to (3.5) the Poisson's kinematic equation

$$\alpha_1^{\bullet} = \alpha_2 r - \alpha_3 q \quad (1\ 2\ 3) \tag{3.8}$$

where $(\alpha_i\ (i = 1, 2, 3)$ are direction cosines of the fixed vector S with moving axes.

Problem. Determine the control moment $\mathbf{M}$ which would make the body asymptotically approach such position that vector $\mathbf{H}$ attached to the body would assume the direction of the fixed vector $\mathbf{S}$. Let us consider moment

$$\mathbf{M} = \boldsymbol{\mu} - \lambda (\mathbf{z} \times \mathbf{H}_0) + {}^1/_2\, \mathrm{grad}_{\alpha_i} U \tag{3.9}$$

where $\mathbf{z}$ is a vector with components z_1, z_2 and z_3.

Equations (3.5), (3.9) and (3.8) admit the particular solution

$$\boldsymbol{\omega} = \lambda \mathbf{H}_0, \quad \mathbf{H}_0 = \mathbf{S}_0 \tag{3.10}$$

for any z_i $(i = 1, 2, 3)$, that satisfy conditions (3.6) and (3.7).

For $\boldsymbol{\omega} = \lambda \mathbf{H}_0$ Eqs. (3.6) assume the form

$$z_1^{\bullet} = \lambda (\beta_3 z_2 - \beta_2 z_3) \quad (1\ 2\ 3) \tag{3.11}$$

Equations (3.11) admit two first integrals

$$z_1^2 + z_2^2 + z_3^2 = h_1^2 = \mathrm{const}, \quad \beta_1 z_1 + \beta_2 z_2 + \beta_3 z_3 = h = \mathrm{const} \tag{3.12}$$

The general solution of Eqs-(3.11) is of the form

$$z_1 = a_1 \cos \lambda t + b_1 \sin \lambda t + h\beta_1 \tag{3.13}$$

$$z_2 = \frac{1}{1-\beta_1^2} [(\beta_3 b_1 - \beta_1\beta_2 a_1) \cos \lambda t - (\beta_3 a_1 + \beta_1\beta_2 b_1) \sin \lambda t] + h\beta_2$$

$$z_3 = -\frac{1}{1-\beta_1^2} [(\beta_2 b_1 + \beta_1\beta_3 a_1) \cos \lambda t - (\beta_2 a_1 - \beta_1\beta_3 b_1) \sin \lambda t] + h\beta_3$$

$$a_1^2 + b_1^2 = (1 - \beta_1^2)(h_1^2 - h^2), \quad h_1 \geqslant h$$

Let the initial conditions $z_i(0)$ $(i = 1, 2, 3)$ for system (3.11) be such that

$$h = h_1 \tag{3.14}$$

Then $(a_1 = b_1 = 0)$ the solution of system (3.11) assumes the form

$$z_1 = h\beta_1, \quad z_2 = h\beta_2, \quad z_3 = h\beta_3 \tag{3.15}$$

and moment (3.9) becomes

$$\mathbf{M} = \boldsymbol{\mu} + {}^1/_2\, \mathrm{grad}_{\alpha_i} U \tag{3.16}$$

Let us restrict our analysis to the case of (3.14), (3.15). Using for the variation of variables the notation

$$p_1 = p - \lambda\beta_1, \quad q_1 = q - \lambda\beta_2, \quad r_1 = r - \lambda\beta_3, \quad \alpha_i - \beta_i = \delta_i,$$

$$z_i - h\beta_i = \eta_i \quad (i = 1, 2, 3)$$

we formulate the equations of perturbed motion

$$A p_1^{\bullet} = h(\beta_2 r_1 - \beta_3 q_1) + \lambda(\beta_3\eta_2 - \beta_2\eta_3) + \eta_2 r_1 - \eta_3 q_1 + \alpha\delta_1 + \mu_1 \quad (1\ 2\ 3) \tag{3.17}$$
$$\delta_1^{\bullet} = \beta_2 r_1 - \beta_3 q_1 + \lambda(\beta_3\delta_2 - \beta_2\delta_3) + \delta_2 r_1 - \delta_3 q_1 \quad (1\ 2\ 3)$$
$$\eta_1^{\bullet} = h(\beta_2 r_1 - \beta_3 q_1) + \lambda(\beta_3\eta_2 - \beta_2\eta_3) + \eta_2 r_1 - \eta_3 q_1 \quad (1\ 2\ 3)$$

We denote by $\mathbf{x}$ the vector with components $\{p_1, q_1, r_1, \delta_1, \delta_2, \delta_3, \eta_1, \eta_2, \eta_3)$ and by $\mathbf{y}$ the vector with components $\{p_1, q_1, r_1, \delta_1, \delta_2, \delta_3\}$ and examine function

$$2V = Ap_1^2 + Bq_1^2 + Cr_1^2 - 2\lambda(Ap_1\delta_1 + Bq_1\delta_2 + Cr_1\delta_3) + \alpha(\delta_1^2 + \delta_2^2 + \delta_3^2) + (h\delta_1 - \eta_1)^2 + (h\delta_2 - \eta_2)^2 + (h\delta_3 - \eta_3)^2 \tag{3.18}$$

Function (3.18) is $\mathbf{y}$-positive-definite in region [4]

$$0 \leqslant |\mathbf{x}| < +\infty \tag{3.19}$$

if $\alpha > \lambda^2 m_1$, where $m_1 = \max\{A, B, C\}$. Furthermore, for $V(\mathbf{x}) \to \infty$, $|\mathbf{y}| \to \infty$. The total derivative of function (3.18) with respect to time, derived on the basis of equations of perturbed motion (3.17) is of the form

$$V^{\bullet} = -h[(p_1 - \lambda\delta_1)^2 + (q_1 - \lambda\delta_2)^2 + (r_1 - \lambda\delta_3)^2] \tag{3.20}$$

obtained by setting in Eqs. (3.17) $\alpha = \lambda h$ and defining functions μ_i $(i = 1, 2, 3)$ by the equalities

$$\mu_1 = -hp_1 + \lambda(B\beta_3 q_1 - C\beta_2 r_1) + (B - C)q_1 r_1 - \lambda(\beta_3\eta_2 - \beta_2\eta_3) - (\eta_2 r_1 - \eta_3 q_1) \tag{3.21}$$
$$\mu_2 = -hq_1 + \lambda(C\beta_1 r_1 - A\beta_3 p_1) + (C - A)p_1 r_1 - \lambda(\beta_1\eta_3 - \beta_3\eta_1) - (\eta_3 p_1 - \eta_1 r_1)$$
$$\mu_3 = -hr_1 + \lambda(A\beta_2 p_1 - B\beta_1 q_1) + (A - B)p_1 q_1 - \lambda(\beta_2\eta_1 - \beta_1\eta_2) - (\eta_1 q_1 - \eta_2 p_1)$$

Let us consider the set $\{\mathbf{x} : \mathbf{y} = 0\}$. It constitutes the aggregate of all solutions of the linear equations

$$\eta_1^{\bullet} = \lambda(\beta_3\eta_2 - \beta_2\eta_3) \quad (1\ 2\ 3)$$

i.e. it is invariant. The manifold E of points at which (3.20) becomes zero is of the form $p_1 = \lambda\delta_1$, $q_1 = \lambda\delta_2$, $r_1 = \lambda\delta_3$ and η_i are arbitrary $(i = 1, 2, 3)$.

For $p_1 = \lambda\delta_1 = \lambda l_1$, $q_1 = \lambda\delta_2 = \lambda l_2$, $r_1 = \lambda\delta_3 = \lambda l_3$ the first six equations (3.17) are identically satisfied and the remaining three assume the form

$$\eta_1^{\bullet} = \lambda[(\beta_3 + l_3)\eta_2 - (\beta_2 + l_2)\eta_3] + \lambda h(\beta_2 l_3 - \beta_3 l_2) \quad (1\ 2\ 3) \tag{3.22}$$

Equations (3.22) admit the first integral

$$\eta_1(\beta_1 + l_1) + \eta_2(\beta_2 + l_2) + \eta_3(\beta_3 + l_3) = n = \text{const} \tag{3.23}$$

The general solution of Eqs. (3.22) is of the form

$$\eta_i = \varphi_i(t) + n(\beta_i + l_i) + hl_i \quad (i = 1, 2, 3) \tag{3.24}$$

where $\varphi_i(t)$ are periodic functions of period 2π, $\varphi_i(t) + n(\beta_i + l_i)$ is the general solution of the homogeneous system, and hl_i is the particular solution of the nonhomogeneous system (3.22).

The substitution of solution (3.24) into the integral (3.23) with allowance for the relationship

$$l_1^2 + l_2^2 + l_3^2 + 2(\beta_1 l_1 + \beta_2 l_2 + \beta_3 l_3) = 0$$

yields $l_1^2 + l_2^2 + l_3^2 = 0$, i.e. $l_i = 0$. Thus, we obtain that the set $E\setminus\{\mathbf{x} : \mathbf{y} = 0\}$ does not contain complete motions of system (3.17) and, consequently, for $h > \lambda m_1$ ($m_1 = \max\{A, B, C\}$) the unperturbed motion $\mathbf{x} = 0$ is asymptotically $\mathbf{y}$-stable as a whole [5].

Theorem 3. If the initial conditions $z_i(0)$ $(i = 1, 2, 3)$ for the linear system (3.11) are selected in accordance with (3.12) and (3.14), and the conditions $h > \lambda m_1$ are satisfied, the solid body subjected to moment

$$\mathbf{M} = \boldsymbol{\mu} + {}^1/_2 \operatorname{grad}_{\alpha_i} U$$

where $\boldsymbol{\mu}$ is determined by (3.21), either performs the motion

$$\boldsymbol{\omega} = \lambda \mathbf{H}_0, \quad \mathbf{H}_0 = \mathbf{S}_0$$

or asymptotically tends to such motion, and the motion (3.24) is asymptotically stable according to Liapunov.

4. Let the gyrostat move about the center of mass in the presence of forces of gravity with the fixed point O of the gyrostat located at distance R from the center of pull O_1.

Let us investigate moment

$$\mathbf{M} = \boldsymbol{\mu}^\circ - \nu(\mathbf{H}_0 \times \Theta \mathbf{S}_0) + {}^1/_2 \operatorname{grad}_{\alpha_i} U \tag{4.1}$$

where Θ is the tensor of the girostat inertia about the fixed point O.

If condition (3.14) is satisfied, the equations of perturbed motion with allowance for moments of gravitational forces are of the form

$$\begin{aligned} Ap_1^{\cdot} &= h(\beta_2 r_1 - \beta_3 q_1) + \lambda(\beta_3\eta_2 - \beta_2\eta_3) + \eta_2 r_1 - \eta_3 q_1 + \\ &\quad \nu(C_3 - C_2)\delta_2\delta_3 + \nu(C_3\beta_3\delta_2 - C_2\beta_2\delta_3) + \alpha\delta_1 + \mu_1^\circ \quad (1\ 2\ 3) \\ \delta_1^{\cdot} &= \beta_2 r_1 - \beta_3 q_1 + \lambda(\beta_3\delta_2 - \beta_2\delta_3) + \delta_2 r_1 - \delta_3 q_1 \quad (1\ 2\ 3) \\ \eta_1^{\cdot} &= h(\beta_2 r_1 - \beta_3 q_1) + \lambda(\beta_3\eta_2 - \beta_2\eta_3) + (\eta_2 r_1 - \eta_3 q_1) \quad (1\ 2\ 3) \end{aligned} \tag{4.2}$$

Function

$$\begin{aligned} 2V &= Ap_1^2 + Bq_1^2 + Cr_1^2 - 2\lambda(Ap_1\delta_1 + Bq_1\delta_2 + Cr_1\delta_3) + \\ &\quad (\alpha + \nu C_1)\delta_1^2 + (\alpha + \nu C_2)\delta_2^2 + (\alpha + \nu C_3)\delta_3^2 + (h\delta_1 - \\ &\quad \eta_1)^2 + (h\delta_2 - \eta_2)^2 + (h\delta_3 - \eta_3)^2 \end{aligned} \tag{4.3}$$

for

$$h > \max\left\{\frac{(\lambda^2 - \nu)A + \nu J_1}{\lambda}, \frac{(\lambda^2 - \nu)B + \nu J_2}{\lambda}, \frac{(\lambda^2 - \nu)C + \nu J_2}{\lambda}\right\} \tag{4.4}$$

is a $\mathbf{y}$-positive-definite function. Let us set

$$\begin{aligned} \mu_1^\circ &= \mu_1 - \nu(C_2 + C_3)(\beta_3\delta_2 - \beta_2\delta_3), \quad \mu_2^\circ = \mu_2 - \nu(C_1 + \\ & C_3)(\beta_1\delta_3 - \beta_3\delta_1), \quad \mu_3^\circ = \mu_3 - \nu(C_1 + C_2)(\beta_2\delta_1 - \beta_1\delta_2) \end{aligned} \tag{4.5}$$

where μ_i $(i = 1, 2, 3)$ are functions defined by (3.21). The total derivative of function (4.3) with respect to time, based on equations of perturbed motion (4.2) with allowance for equalities (3.21) and (4.5), is of the form

$$V^{\cdot} = -h\,[(p_1 - \lambda\delta_1)^2 + (q_1 - \lambda\delta_2)^2 + (r_1 - \lambda\delta_3)^2]$$

The set $\{\mathbf{x} : \mathbf{y} = 0\}$ is invariant and the set $E \setminus \{\mathbf{x} : \mathbf{y} = 0\}$ does not contain complete trajectories of system (4.2) (the proof of this is similar to [3]). Consequently, if conditions (3.14) and (4.4) are satisfied, the unperturbed motion $\mathbf{x} = 0$ is asymptotically $\mathbf{y}$-stable as a whole [5].

Theorem 4. If the initial conditions $z_i(0)$ $(i = 1, 2, 3)$ of the linear system (3.11) are selected in accordance with (3.12) and (3.14), and condition (4.4) is satisfied, the solid body sunjected to moment (4.1), where $\boldsymbol{\mu}^\circ$ is determined by equalities (3.21) and (4.5) in the presence of gravitational forces either performs the motion

$$\boldsymbol{\omega} = \lambda \mathbf{H}_0, \quad \mathbf{H}_0 = \mathbf{S}_0 \tag{4.6}$$

or asymptotically tends to such motion. Motion (4.6) is asymptotically stable according to Liapunov.

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ON A CLASS OF PERIODIC MOTIONS OF A SOLID BODY ABOUT A FIXED POINT

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Existence of a new set of periodic solutions of the problem of a heavy solid body motion about a fixed point is proved by the small parameter method of Poincaré. It is assumed that the body does not greatly differ from a body with a dynamic symmetry axis, and that the constant of integration of the moment of momentum is fairly small.

Let us consider the motion of a heavy solid body about a fixed point. The equation of motion of this problem can be reduced to a fourth order system describing the motion of a fictitious material point in a plane, by using the cyclic integral $\partial T / \partial\psi^{\cdot} = f$, where